

TODA HIERARCHY, HURWITZ NUMBERS AND CONFORMAL DYNAMICS

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ABSTRACT. We explicitly construct the series expansion for a certain class of solutions to the 2D Toda hierarchy in the zero dispersion limit, which we call symmetric solutions. We express the Taylor coefficients through some universal combinatorial constants and find recurrence relations for them. These results are used to obtain new formulas for the genus 0 double Hurwitz numbers. They can also serve as a starting point for a constructive approach to the Riemann mapping problem and the inverse potential problem in 2D.

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1. INTRODUCTION

The dispersionless 2D Toda lattice (2DTL) hierarchy was introduced by Takasaki and Takebe [28, 29]. It can be represented in two equivalent ways: in the Lax-Sato form or in the Hirota form. In this paper we use the latter formulation. The dispersionless 2DTL hierarchy is an infinite system of differential equations

$$(1) \quad (z - \xi) \exp(D(z)D(\xi)F) = z \exp(-\partial_0 D(z)F) - \xi \exp(-\partial_0 D(\xi)F),$$

$$(2) \quad (\bar{z} - \bar{\xi}) \exp(\bar{D}(\bar{z})\bar{D}(\bar{\xi})F) = \bar{z} \exp(-\partial_0 \bar{D}(\bar{z})F) - \bar{\xi} \exp(-\partial_0 \bar{D}(\bar{\xi})F),$$

$$(3) \quad 1 - \exp(-D(z)\bar{D}(\bar{\xi})F) = \frac{1}{z\xi} \exp(\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F)$$

for the function $F = F(t_0, \mathbf{t}, \bar{\mathbf{t}})$, where $\mathbf{t} = \{t_1, t_2, \dots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \dots\}$ are two infinite sets of time variables and

$$(4) \quad D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k.$$

Hereafter we abbreviate $\partial_k = \partial/\partial t_k$, $\bar{\partial}_k = \partial/\partial \bar{t}_k$. Differential equations of the hierarchy are obtained by expanding equations (1)-(3) in powers of z, ξ . These equations are known to be connected with different branches of mathematics and mathematical physics.

Solutions to equations (1)-(3) such that $\partial_k F|_{t_0} = \bar{\partial}_k F|_{t_0} = 0$ for all $k \geq 1$ form an especially important class. (By $g|_{t_0}(t_0)$ we denote the restriction of a function $g(t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots)$ to the line $t_1 = \bar{t}_1 = t_2 = \bar{t}_2 = \dots = 0$.) We call them *symmetric solutions*. This class contains, in particular, the $c = 1$ string solution [2, 3, 8, 30].

In Section 2 we describe all formal symmetric solutions of the dispersionless 2DTL hierarchy in the form of a Taylor series. We prove that they are fully defined by the restriction $F|_{t_0}$ (a function of one variable). The Taylor coefficients are expressed through some universal combinatorial constants $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$ which depend on two Young diagrams $\Delta, \bar{\Delta}$ and two sequences of natural numbers $\{s_i\}, \{r_i\}$. Moreover, we find recurrence formulas for $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$. Our method is an extension of the method developed in [22, 23] for the formal $c = 1$ string solution. Convergence of the Taylor series for this case was investigated in [11].

In Section 3 we apply these results to the double Hurwitz numbers for coverings of genus 0. This application is based on the fact that the generating function for connected double Hurwitz numbers of genus 0 is a symmetric solution of the dispersionless 2DTL hierarchy, which follows from Okounkov's result [25] and its dispersionless limit discussed by Takasaki [27].

The connected genus 0 double Hurwitz number $H_0(\Delta|\bar{\Delta})$ is the number of non-equivalent rational functions with pre-images of 0 and ∞ of fixed topological types given by Young diagrams $\Delta = [k_1, \dots, k_\ell]$ and $\bar{\Delta} = [\bar{k}_1, \dots, \bar{k}_{\bar{\ell}}]$ (with the condition on degrees $|\Delta| = |\bar{\Delta}|$) and some fixed simple critical values (finite and non-zero). For rational functions of degree d with simple finite critical values this number was found by Hurwitz [9]:

$$H_0([2, \underbrace{1, \dots, 1}_{d-2} | [k_1, \dots, k_n]) = H_0([1, \dots, 1] | [k_1, \dots, k_n]) = \frac{(d+n-2)!}{\sigma([k_1, \dots, k_n])} \prod_{i=1}^n \frac{k_i^{k_i}}{k_i!} d^{n-3}$$

where $\sum_i k_i = d$ and $\sigma(\Delta)$ is order of the automorphism group of rows for the Young diagram Δ .

Further results are based on the moduli space methods [16]. In [17] it was found, in particular, that

$$H_0([n]|\Delta) = \frac{(m-1)!}{\sigma(\Delta)} n^{m-2}.$$

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Some formulas for $H_0([n_1, n_2]|\Delta)$ and $H_0([n_1, n_2, n_3]|\Delta)$ were found in [6, 26]. It was also proved that the Hurwitz numbers $H_0([k_1, \dots, k_n][\bar{k}_1, \dots, \bar{k}_{\bar{n}}])$ depend piecewise polynomially on $k_1, \dots, k_n, \bar{k}_1, \dots, \bar{k}_{\bar{n}}$ [6]. The domains of polynomiality (chambers) were characterized and relations between polynomials in neighboring chambers were found [26].

In Section 3, by comparing the generating function for the genus 0 double Hurwitz numbers with the formulas for symmetric solutions, we give a representation of the Hurwitz numbers through the combinatorial coefficients $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$. These formulas appear to be new and useful. In particular, they make explicit some general properties of the genus 0 double Hurwitz numbers (for example, the piecewise polynomiality). Moreover, they are well-suited for direct calculations with the help of the computer and allow one to obtain closed formulas for $H_0([k_1, \dots, k_n][\bar{k}_1, \dots, \bar{k}_{\bar{n}}])$ for any n, \bar{n} . We illustrate this method by two examples: $H_0([n][k_1, \dots, k_m])$ and $H_0([k_1, k_2][\bar{k}_1, \bar{k}_2])$, where the calculations can be done by hands. Other consequences of our general formulas will be discussed elsewhere.

Another application, addressed in Section 4, is the class of problems which we call conformal dynamics. In papers [19, 33, 12, 18] it was shown that some problems of complex analysis in 2D, such as conformal mapping, Dirichlet boundary value problem and 2D inverse potential problem have a hidden integrable structure, which, for simply-connected domains, is the dispersionless 2DTL hierarchy. In a more general case it is the universal Whitham hierarchy introduced by Krichever in [13, 14]. The hierarchical times t_k are suitably defined harmonic moments of the domain and their complex conjugates, with t_0 being proportional to the area of the domain. One can construct the dispersionless tau-function $F = F(t_0, \{t_k\}, \{\bar{t}_k\})$ which contains all the information about the conformal bijection of any domain with given moments to the unit disk. The function F satisfies the dispersionless version of the Hirota equations for the 2DTL hierarchy given by (1)-(3).

Further, in [35] it was argued that any non-degenerate solution of the hierarchy, with certain reality conditions imposed, can be given a similar geometric meaning. Such solutions are parameterized by a function $\sigma(z, \bar{z})$ of two variables which has the meaning of a density, or conformal metric, in the complex plane. The moments should be now defined as integrals of powers of z with this density. The integral representation for the dispersionless tau-function also changes accordingly but the formulas which express the conformal map through its second order derivatives do not depend on σ . In other words, the Toda dynamics encodes the shape dependence of the conformal mapping, which we call the *conformal dynamics*. In the context of the conformal dynamics, the symmetric solutions correspond to the axially symmetric functions σ (i.e., depending only on $|z|^2$).

Some important examples of conformal dynamics on symmetric background are considered in Section 5. Using the approach of Section 2, one can write explicit formulas for the corresponding symmetric solutions of the dispersionless 2DTL hierarchy. The corresponding conformal dynamics provides an effectivization of the Riemann mapping theorem. Namely, for any domain characterized by its moments, our formulas allow one to find the conformal map from this domain to the unit disk with any given precision.

2. FORMAL SYMMETRIC SOLUTIONS OF DISPERSIONLESS 2D TODA HIERARCHY

Let $\Delta = [\mu_1, \mu_2, \dots, \mu_\ell]$ be the Young diagram with $\ell = \ell(\Delta)$ rows of non-zero lengths $\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell > 0$, and similarly for $\bar{\Delta} = [\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_{\bar{\ell}}]$, with $\bar{\ell} = \ell(\bar{\Delta})$. We identify Δ with the partition of the number $|\Delta| := \mu_1 + \dots + \mu_\ell$ into the ℓ non-zero parts μ_i . Another convenient notation is $\Delta = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots)$, which means that exactly m_i parts of the partition Δ have length i : $m_i = \text{card} \{j : \mu_j = i\}$. Put $\sigma(\Delta) = m_1! \dots m_r!$ and $\rho(\Delta) = \mu_1 \dots \mu_{\ell(\Delta)}$.

Given the two sets of time variables, $\mathbf{t} = \{t_1, t_2, \dots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \dots\}$ as before, set $t_\Delta = t_{\mu_1} \dots t_{\mu_\ell}$, $\bar{t}_\Delta = \bar{t}_{\mu_1} \dots \bar{t}_{\mu_\ell}$. For the empty diagram we put $t_\emptyset = \bar{t}_\emptyset = 1$. We say that a Taylor series of the form

$$(5) \quad F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{\Delta, \bar{\Delta}} F(\Delta | \bar{\Delta} | t_0) t_\Delta \bar{t}_{\bar{\Delta}},$$

where the sum is over all pairs of Young diagrams including empty ones, is a *formal solution* to the dispersionless 2DTL hierarchy if its substitution to equations (1)-(3) gives the identity. In a more explicit but less compact notation the series (5) reads

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = F(t_0) + \sum_{\substack{\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell \\ \bar{\mu}_1 \geq \bar{\mu}_2 \geq \dots \geq \bar{\mu}_{\bar{\ell}}}} F(\mu_1, \mu_2, \dots, \mu_\ell | \bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_{\bar{\ell}} | t_0) t_{\mu_1} t_{\mu_2} \dots t_{\mu_\ell} \bar{t}_{\bar{\mu}_1} \bar{t}_{\bar{\mu}_2} \dots \bar{t}_{\bar{\mu}_{\bar{\ell}}}.$$

Here $F(t_0) = F(\emptyset | \emptyset | t_0)$.

Remark. In this section the bar does not mean the complex conjugation and t_k, \bar{t}_k are regarded as arbitrary formal variables.

2.1. Taylor expansion of symmetric solutions. Let $g|_{t_0}(t_0)$ denote the restriction of a function $g(t_0, t_1, \bar{t}_1, t_2, \bar{t}_2, \dots)$ to the line $t_1 = \bar{t}_1 = t_2 = \bar{t}_2 = \dots = 0$. Formal solutions such that $\partial_k F|_{t_0} = \bar{\partial}_k F|_{t_0} = 0$ for all $k \geq 1$ will be called *symmetric*.

Our first goal is to prove that symmetric formal solutions are fully determined by the function $F|_{t_0}(t_0) := F(t_0)$ which can be an arbitrary twice differentiable function of t_0 . Moreover, we prove that $\Phi(\Delta | \bar{\Delta} | t_0)$ is a differential polynomial in $f(t_0) = \exp(F|_{t_0}''(t_0))$ (i.e., a polynomial in $f(t_0), f'(t_0), f''(t_0), \dots$) with universal coefficients which will be found. For a particular function $f(t_0)$ they were found in [22]. In the general case the arguments are similar. We split the proof into several lemmas.

Lemma-Definition 2.1. ([22], Lemma 3.3) *For any formal solution F the following relation holds*

$$\partial_i \partial_j F = \sum_{\substack{m \geq 0 \\ p_1 + \dots + p_m = i+j}} \frac{i^j}{p_1 \dots p_m} T_{ij}(p_1, \dots, p_m) \partial_0 \partial_{p_1} F \dots \partial_0 \partial_{p_m} F,$$

where

$$T_{ij}(p_1, \dots, p_m) = \sum_{\substack{k \geq 0, n_i \geq 0 \\ n_1 + \dots + n_k = m}} \frac{(-1)^{m+1}}{k n_1! \dots n_k!} P_{ij} \left(\sum_{i=1}^{n_1} p_i, \sum_{i=n_1+1}^{n_1+n_2} p_i, \dots, \sum_{i=n_1+\dots+n_{k-1}+1}^m p_i \right)$$

and $P_{ij}(r_1, \dots, r_m)$ is the number of sequences of positive integers $(i_1, \dots, i_m), (j_1, \dots, j_m)$ such that $i_1 + \dots + i_m = i$, $j_1 + \dots + j_m = j$ and $r_k = i_k + j_k$.

An analog of this lemma for the dispersionless KP hierarchy can be found in [24]. Here and below all indices and variables with indices like $i, p, s, \ell, n, r, a, b$ with or without bar are positive integers.

By induction one can prove

Lemma-Definition 2.2. ([22], Lemma 3.4.) *For any formal solution F the following relation holds*

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} F = \sum_{m=1}^{\infty} \left(\sum_{\substack{s_1 + \cdots + s_m = i_1 + \cdots + i_k \\ \ell_1 + \cdots + \ell_m = m + k - 2}} \frac{i_1 \cdots i_k}{s_1 \cdots s_m} T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{smallmatrix} \right) \partial_0^{\ell_1} \partial_{s_1} F \cdots \partial_0^{\ell_m} \partial_{s_m} F \right),$$

where

$$T_{i_1 i_2} \left(\begin{smallmatrix} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{smallmatrix} \right) = \begin{cases} T_{i_1 i_2}(s_1, \dots, s_m), & \text{if } \ell_1 = \dots = \ell_m = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \cdots s_m \\ \ell_1 \cdots \ell_m \end{smallmatrix} \right) &= \sum_{1 \leq i \leq j \leq m} \frac{\ell!}{(\ell_i - 1)! \cdots (\ell_j - 1)!} \\ &\times T_{i_1 \dots i_{k-1}} \left(\begin{smallmatrix} s_1 \cdots s_{i-1} s s_{j+1} \cdots s_m \\ \ell_1 \cdots \ell_{i-1} \ell \ell_{j+1} \cdots \ell_m \end{smallmatrix} \right) T_{s, i_k}(s_i, s_{i+1}, \dots, s_j), \end{aligned}$$

with

$$s = s_i + s_{i+1} + \cdots + s_j - i_k > 0, \quad \ell = (\ell_i - 1) + \cdots + (\ell_j - 1) > 0.$$

From now on we consider only symmetric formal solutions F . Put $f := \exp(F|_{t_0}'')$.

Lemma 2.1. *If F is a symmetric formal solution, then*

$$\partial_i \bar{\partial}_j F|_{t_0} = \begin{cases} 0 & \text{for } i \neq j, \\ if^i & \text{for } i = j. \end{cases}$$

Proof. From $\partial_0 \partial_k F|_{t_0} = \partial_0 \bar{\partial}_k F|_{t_0} = 0$ for $k > 0$ it follows that

$$\exp(\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F) \Big|_{t_0} = \exp(\partial_0^2 F|_{t_0}) = \exp(F|_{t_0}'') = f.$$

Moreover, from $1 - e^{-D(z)\bar{D}(\bar{\xi})F} = z^{-1}\bar{\xi}^{-1}e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F}$ we have

$$-D(z)\bar{D}(\bar{\xi})F|_{t_0} = \log(1 - z^{-1}\bar{\xi}^{-1}f) = -\sum_{k=1}^{\infty} \frac{1}{k} z^{-k} \bar{\xi}^{-k} f^k.$$

Therefore, $\partial_i \bar{\partial}_j F|_{t_0} = 0$ for $i \neq j$ and $\partial_i \bar{\partial}_i F|_{t_0} = if^i$. □

Lemma 2.2. *The following relations hold*

$$\partial_i \bar{\partial}_{i_1} \cdots \bar{\partial}_{i_k} F|_{t_0} = \bar{\partial}_i \partial_{i_1} \cdots \partial_{i_k} F|_{t_0} = \begin{cases} 0 & \text{if } i_1 + \cdots + i_k \neq i \\ i_1 \cdots i_k \partial_0^{k-1}(f^i) & \text{if } i_1 + \cdots + i_k = i. \end{cases}$$

Proof. The differentials ∂ and $\bar{\partial}$ enter the Toda equations in a symmetric way. This gives the first equality. Moreover, according to Lemma-Definition 2.2, we have

$$\bar{\partial}_i \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} F = \frac{i_1 \cdots i_k}{i_1 + \cdots + i_k} \partial_0^{k-1} \bar{\partial}_i \partial_{i_1 + \dots + i_k} F$$

$$+ \bar{\partial}_i \sum_{m=2}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ \ell_1+\dots+\ell_m=m+k-2}} \frac{i_1 \dots i_k}{s_1 \dots s_m} T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \dots s_m \\ \ell_1 \dots \ell_m \end{smallmatrix} \right) \partial_0^{\ell_1} \partial_{s_1} F \dots \partial_0^{\ell_m} \partial_{s_m} F.$$

This equality and Lemma 2.1 gives the second equality in the assertion of Lemma 2.2. \square

Given a sequence of natural numbers $\{a_1, \dots, a_v\}$, we say that its representation as a union of non-intersecting non-empty subsequences $\{b_1^j, \dots, b_{n_j}^j\}$ such that $b_1^j + \dots + b_{n_j}^j = s_j$, $\{a_1, \dots, a_v\} = \bigcup_{j=1}^m \{b_1^j, \dots, b_{n_j}^j\}$, is a *partition of $\{a_1, \dots, a_v\}$ of the type $\left(\begin{smallmatrix} s_1 \dots s_m \\ n_1 \dots n_m \end{smallmatrix} \right)$* .

Clearly, the sequence (n_1, \dots, n_m) is a partition of v in the usual sense: $\sum_{i=1}^m n_i = v$.

Lemma-Definition 2.3. *The following relation holds:*

$$\partial_{i_1} \dots \partial_{i_k} \bar{\partial}_{\bar{i}_1} \dots \bar{\partial}_{\bar{i}_k} F \Big|_{t_0} = \sum_{m=1}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k=\bar{i}_1+\dots+\bar{i}_k \\ r_1+\dots+r_m=m+k+\bar{k}-2}} \tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_k \end{smallmatrix} \right)} \left(\begin{smallmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{smallmatrix} \right) \partial_0^{r_1} f^{s_1} \dots \partial_0^{r_m} f^{s_m},$$

where

$$\tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_k \end{smallmatrix} \right)} \left(\begin{smallmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{smallmatrix} \right) = \frac{i_1 \dots i_k \bar{i}_1 \dots \bar{i}_k}{s_1 \dots s_m} \sum T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \dots s_m \\ r_1 - n_1 + 1 \dots r_m - n_m + 1 \end{smallmatrix} \right)$$

and the summation is carried over all partitions of the set $\{\bar{i}_1, \dots, \bar{i}_k\}$ of type $\left(\begin{smallmatrix} s_1 \dots s_m \\ n_1 \dots n_m \end{smallmatrix} \right)$.

Proof. According to Lemma-Definition 2.2,

$$\begin{aligned} & \partial_{i_1} \dots \partial_{i_k} \bar{\partial}_{\bar{i}_1} \dots \bar{\partial}_{\bar{i}_k} F \\ &= \bar{\partial}_{\bar{i}_1} \dots \bar{\partial}_{\bar{i}_k} \left(\sum_{m=1}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ l_1+\dots+l_m=m+k-2}} \frac{i_1 \dots i_k}{s_1 \dots s_m} \cdot T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right) \partial_0^{l_1} \partial_{s_1} F \dots \partial_0^{l_m} \partial_{s_m} F \right) \\ &= \sum_{m=1}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ l_1+\dots+l_m=m+k-2}} \sum \left(\frac{i_1 \dots i_k}{s_1 \dots s_m} T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right) \right. \\ & \quad \left. \times \partial_0^{l_1} \partial_{s_1} \bar{\partial}_{\bar{j}_1^1} \dots \bar{\partial}_{\bar{j}_1^1} F \dots \partial_0^{l_m} \partial_{s_m} \bar{\partial}_{\bar{j}_1^m} \dots \bar{\partial}_{\bar{j}_1^m} F \right), \end{aligned}$$

where the interior summation is carried over all partitions of $\{\bar{i}_1, \dots, \bar{i}_k\}$ of type $\left(\begin{smallmatrix} s_1 \dots s_m \\ n_1 \dots n_m \end{smallmatrix} \right)$. Therefore, according to Lemma 2.2,

$$\begin{aligned} & \partial_{i_1} \dots \partial_{i_k} \bar{\partial}_{\bar{i}_1} \dots \bar{\partial}_{\bar{i}_k} F \\ &= \sum_{m=1}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ l_1+\dots+l_m=m+k-2}} \frac{i_1 \dots i_k \bar{i}_1 \dots \bar{i}_k}{s_1 \dots s_m} \sum \left(T_{i_1 \dots i_k} \left(\begin{smallmatrix} s_1 \dots s_m \\ l_1 \dots l_m \end{smallmatrix} \right) \partial_0^{l_1+n_1-1} (f^{s_1}) \dots \partial_0^{l_m+n_m-1} (f^{s_m}) \right) \\ &= \sum_{m=1}^{\infty} \sum_{\substack{s_1+\dots+s_m=i_1+\dots+i_k \\ r_1+\dots+r_m=m+k+\bar{k}-2}} \frac{i_1 \dots i_k \bar{i}_1 \dots \bar{i}_k}{s_1 \dots s_m} \end{aligned}$$

$$\times \sum \left(T_{i_1 \dots i_k} \left(r_1 - n_1 + 1 \dots r_m - r_m + 1 \right) \partial_0^{r_1}(f^{s_1}) \dots \partial_0^{r_m}(f^{s_m}) \right),$$

where the interior summation is carried over all partitions of the set $\{\bar{i}_1, \dots, \bar{i}_k\}$ of type $\binom{s_1 \dots s_m}{n_1 \dots n_m}$. \square

Lemma-Definition 2.3 implies

Theorem 2.1. *Any symmetric formal solution to the dispersionless 2DTL has the form*

$$(6) \quad F = F(t_0) + \sum_{|\Delta|=|\bar{\Delta}|} \sum_{\substack{s_1+\dots+s_m=|\Delta| \\ r_1+\dots+r_m=m+\ell(\Delta)+\ell(\bar{\Delta})-2}} N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m} \partial_0^{r_1}(f^{s_1}) \dots \partial_0^{r_m}(f^{s_m}) t_{\Delta} t_{\bar{\Delta}},$$

where $f(t_0) = \exp(\partial_0^2 F(t_0))$,

$$(7) \quad N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m} = \frac{1}{\sigma(\Delta)\sigma(\bar{\Delta})} \tilde{N}_{\left(\begin{smallmatrix} \mu_1 \dots \mu_k \\ \bar{\mu}_1 \dots \bar{\mu}_k \end{smallmatrix}\right)} \binom{s_1 \dots s_m}{r_1 \dots r_m}$$

and $[\mu_1, \dots, \mu_\ell] = \Delta$, $[\bar{\mu}_1, \dots, \bar{\mu}_\ell] = \bar{\Delta}$.

Theorem 2.1 implies that the symmetric solutions are parameterized by a function of one variable. The Taylor coefficients in the right hand side of equation (6) are differential polynomials in the function $f(t_0) = e^{F''(t_0)}$ with universal coefficients $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$.

Explicit formulas for the simplest coefficients $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$ are given below.

2.2. Explicit formulas for some coefficients.

Lemma 2.3. *Let $\Delta, \bar{\Delta}$ be two Young diagrams such that $|\Delta| = |\bar{\Delta}| = d$. Then*

$$N_{(\Delta|\bar{\Delta})} \binom{|\Delta|}{\ell(\Delta) + \ell(\bar{\Delta}) - 2} = \frac{\rho(\Delta)\rho(\bar{\Delta})}{d\sigma(\Delta)\sigma(\bar{\Delta})}$$

and $N_{(\Delta|\bar{\Delta})} \binom{s}{r} = 0$ otherwise.

Proof. According to our definitions $T_{ij} \binom{i+j}{1} = T_{ij}(i+j) = P_{ij}(i+j) = 1$ and $T_{ij} \binom{s}{l} = 0$ in other cases. Therefore,

$$T_{i_1 \dots i_k} \binom{i_1 + \dots + i_k}{k-1} = T_{i_1 \dots i_{k-1}} \binom{i_1 + \dots + i_{k-1}}{k-2} T_{(i_1+\dots+i_{k-1})i_k}(i_1 + \dots + i_k) = 1$$

and $T_{i_1 \dots i_k} \binom{s}{l} = 0$ in other cases. Let the sequences $(i_1, \dots, i_{\ell(\Delta)})$, $(\bar{i}_1, \dots, \bar{i}_{\ell(\bar{\Delta})})$ be any permutations of $[\mu_1, \dots, \mu_\ell]$, $[\bar{\mu}_1, \dots, \bar{\mu}_\ell]$ respectively, where $[\mu_1, \dots, \mu_\ell] = \Delta$, $[\bar{\mu}_1, \dots, \bar{\mu}_\ell] = \bar{\Delta}$. Then

$$\tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_k \end{smallmatrix}\right)} \binom{|\Delta|}{\ell(\Delta) + \ell(\bar{\Delta}) - 2} = \frac{\rho(\Delta)\rho(\bar{\Delta})}{|\Delta|} T_{i_1 \dots i_k} \binom{|\Delta|}{\ell(\Delta) - 1} = \frac{\rho(\Delta)\rho(\bar{\Delta})}{|\Delta|}$$

and $\tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_k \end{smallmatrix}\right)} \binom{s}{r} = 0$ in other cases. Hence

$$N_{(\Delta|\bar{\Delta})} \left(\ell(\Delta) + \ell(\bar{\Delta}) - 2 \right) = \frac{1}{\sigma(\Delta)\sigma(\bar{\Delta})} \tilde{N}_{\left(\begin{smallmatrix} i_1 \dots i_k \\ \bar{i}_1 \dots \bar{i}_k \end{smallmatrix}\right)} \left(\ell(\Delta) + \ell(\bar{\Delta}) - 2 \right) = \frac{\rho(\Delta)\rho(\bar{\Delta})}{|\Delta|\sigma(\Delta)\sigma(\bar{\Delta})}$$

and $N_{(\Delta|\bar{\Delta})} \binom{s}{r} = 0$ otherwise. \square

Lemma 2.4.

$$\tilde{N}_{\left(\begin{smallmatrix} i_1 i_2 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{\bar{i}_1 \bar{i}_2}{11} = \tilde{N}_{\left(\begin{smallmatrix} i_2 i_1 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{\bar{i}_2 \bar{i}_1}{11} = -\frac{i_1 i_2}{2\sigma([i_1, i_2])\sigma([\bar{i}_1, \bar{i}_2])} \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\}$$

and $\tilde{N}_{\left(\begin{smallmatrix} i_1 i_2 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{s_1 s_2}{r_1 r_2} = 0$ otherwise.

Proof. The number $P_{ij}(r_1, r_2)$ is the number of solutions to the system of equation $i_1 + i_2 = i$, $j_1 + j_2 = j$, $i_1 + j_1 = r_1$, $i_2 + j_2 = r_2$ in positive integer numbers i_1, i_2, j_1, j_2 . This system is equivalent to the system $i_1 = r_1 - j_1$, $i_2 = i + j_1 - r_1$, $j_2 = j - j_1$, $i + j = r_1 + r_2$. Hence $P_{ij}(r_1, r_2) = 0$ if $i + j \neq r_1 + r_2$.

In other cases the number of solutions is the number of positive j_1 such that $r_1 - j_1 > 0$, $i + j_1 - r_1 > 0$, $j - j_1 > 0$ or $j_1 < r_1$, $j_1 < j$, $j_1 > r_1 - i$. By symmetry in pairs i, j and r_1, r_2 it is enough to consider the case $j \leq i$ and $r_1 \leq r_2$. In this case the number of solutions is $\min\{r_1, j\} - 1$. Thus $P_{ij}(r_1, r_2) = \min\{i, j, r_1, r_2\} - 1$.

By our definition

$$T_{i_1, i_2} \binom{s_1 s_2}{11} = T_{ij}(s_1, s_2) = -\frac{1}{2} \left(P_{ij}(s_1 + s_2) + \frac{1}{2} P_{ij}(s_1, s_2) \right) = -\frac{1}{2} \min\{i, j, s_1, s_2\}$$

if $i + j = s_1 + s_2$ and $T_{i_1, i_2} \binom{s_1 s_2}{l_1 l_2} = 0$ in other cases.

By Lemma-Definition 2.3 we have

$$\tilde{N}_{\left(\begin{smallmatrix} i_1 i_2 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{s_1 s_2}{r_1 r_2} = \frac{i_1 i_2 \bar{i}_1 \bar{i}_2}{s_1 s_2} \sum T_{i_1 i_2} \binom{s_1 \quad s_2}{r_1 - n_1 + 1 \quad r_2 - n_2 + 1},$$

where the summation is carried over all partitions of the set $\{\bar{i}_1, \bar{i}_2\}$ of type $\binom{s_1 s_2}{n_1 n_2}$. Such partitions exist only if either $\bar{i}_1 = s_1$, $\bar{i}_2 = s_2$ or $\bar{i}_1 = s_2$, $\bar{i}_2 = s_1$. Therefore,

$$\tilde{N}_{\left(\begin{smallmatrix} i_1 i_2 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{\bar{i}_1 \bar{i}_2}{11} = \tilde{N}_{\left(\begin{smallmatrix} i_2 i_1 \\ \bar{i}_1 \bar{i}_2 \end{smallmatrix}\right)} \binom{\bar{i}_2 \bar{i}_1}{11} = \frac{i_1 i_2 \bar{i}_1 \bar{i}_2}{\bar{i}_1 \bar{i}_2} T_{i_1 i_2} \binom{\bar{i}_1 \quad \bar{i}_2}{1 \quad 1} = -\frac{i_1 i_2}{2} \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\}$$

and 0 in other cases. Finally we obtain

$$N_{[i_1 i_2][\bar{i}_1 \bar{i}_2]} \binom{\bar{i}_1 \bar{i}_2}{11} = N_{[i_1 i_2][\bar{i}_1 \bar{i}_2]} \binom{\bar{i}_2 \bar{i}_1}{11} = -\frac{i_1 i_2}{2\sigma([i_1, i_2])\sigma([\bar{i}_1, \bar{i}_2])} (\min\{i_1, i_2, \bar{i}_1, \bar{i}_2\})$$

and $N_{[i_1 i_2][\bar{i}_1 \bar{i}_2]} \binom{s_1 s_2}{r_1 r_2} = 0$ otherwise. \square

3. DOUBLE HURWITZ NUMBERS

3.1. Generating function for double Hurwitz numbers of genus 0. Given a meromorphic function $\varphi : \Omega \rightarrow \hat{\mathbb{C}}$ of degree d on a connected Riemann surface Ω , one can associate with each point $p \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ a Young diagram $\Delta = \Delta(\varphi, p) = [\mu_1, \dots, \mu_\ell]$ such that $|\Delta| = \deg \varphi = d$ and μ_i equals the degree of the map φ at the point p^i of the complete pre-image $\varphi^{-1}(p) = \{p^1, \dots, p^\ell\}$. Informally speaking, μ_i is the number of sheets of the covering that are glued together at the point p^i lying above p . We say that meromorphic functions $\varphi : \Omega \rightarrow \hat{\mathbb{C}}$ and $\varphi' : \Omega' \rightarrow \hat{\mathbb{C}}$ are equivalent if there exists a biholomorphic map $f : \Omega \rightarrow \Omega'$ such that $\varphi = \varphi' f'$.

Fix different points p_1, \dots, p_k and Young diagrams $\Delta_1, \dots, \Delta_k$ such that $|\Delta_i| = d$ for all $i = 1, \dots, k$. Consider the sum

$$H(\Delta_1, \dots, \Delta_k) = \sum_{\varphi \in \text{Cov}_d(\Omega, \{\Delta_1, \dots, \Delta_k\})} \frac{1}{|\text{Aut}(\varphi)|},$$

where $|\text{Aut}(\varphi)|$ is the order of the automorphism group the covering φ , $\text{Cov}_d(\Omega, \{\Delta_1, \dots, \Delta_k\})$ is the set of biholomorphic equivalence classes of meromorphic functions $\varphi : \Omega \rightarrow \hat{\mathbb{C}}$ such that $\Delta(\varphi, p_i) = \Delta_i$ and $\Delta(\varphi, p) = (1^d)$ at the other points. This sum does not depend on positions of the points p_i . It is called the (*connected*) *Hurwitz number*. See [16] for a review.

The Hurwitz numbers

$$H_{d,l}(\Delta|\bar{\Delta}) = H(\Delta, \bar{\Delta}, \underbrace{(1^{d-2}2^1), \dots, (1^{d-2}2^1)}_l)$$

for functions of degree d with arbitrary ramification types $\Delta, \bar{\Delta}$ at two fixed points and l simple critical values of the type $(1^{d-2}2^1)$ are called *double Hurwitz numbers*. In fact the notation $H_{d,l}(\Delta|\bar{\Delta})$ is redundant because this number is zero unless $d = |\Delta| = |\bar{\Delta}|$. The genus g of Ω and the number l are connected by the Riemann-Hurwitz relation $2g - 2 = l - \ell(\Delta) - \ell(\bar{\Delta})$. Introduce two infinite system of variables $\mathbf{t} = \{t_1, t_2, \dots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \dots\}$, as in the previous section, and consider the generating function

$$(8) \quad \Phi(\beta, Q, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{l \geq 0} \frac{\beta^l}{l!} \sum_{d \geq 1} Q^d \sum_{|\Delta|=|\bar{\Delta}|=d} H_{d,l}(\Delta, \bar{\Delta}) \prod_{i=1}^{\ell(\Delta)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\Delta})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i},$$

where $\Delta = [\mu_1, \dots, \mu_{\ell(\Delta)}]$, $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\ell(\bar{\Delta})}]$.

In order to extract the contribution of genus g surfaces the following trick can be used (see, e.g., [1] for the case of ordinary Hurwitz numbers). Let us rescale $\{t_k\}$, $\{\bar{t}_k\}$ and β by introducing a new parameter \hbar as $t_k \rightarrow t_k/\hbar$, $\beta \rightarrow \hbar\beta$ and consider the modified generating function $\Phi(\hbar; \beta, Q, \mathbf{t}, \bar{\mathbf{t}}) := \hbar^2 \Phi(\hbar\beta, Q, \mathbf{t}/\hbar, \bar{\mathbf{t}}/\hbar)$, then the series (8) having regard to the Riemann-Hurwitz formula acquires the form of the topological expansion

$$(9) \quad \Phi(\hbar; \beta, Q, \mathbf{t}, \bar{\mathbf{t}}) = \sum_{g \geq 0} \hbar^{2g} \Phi_g(\beta, Q, \mathbf{t}, \bar{\mathbf{t}}),$$

where

$$(10) \quad \Phi_g = \sum_{d \geq 1} \sum_{|\Delta|=|\bar{\Delta}|=d} \frac{Q^d \beta^{\ell(\Delta)+\ell(\bar{\Delta})+2g-2}}{(\ell(\Delta)+\ell(\bar{\Delta})+2g-2)!} H_{d,\ell(\Delta)+\ell(\bar{\Delta})+2g-2}(\Delta, \bar{\Delta}) \prod_{i=1}^{\ell(\Delta)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\Delta})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i}$$

counts the connected coverings of genus g . In particular,

$$(11) \quad \Phi_0 = \sum_{d \geq 1} \sum_{|\Delta| = |\bar{\Delta}| = d} \frac{Q^d H_{d, \ell(\Delta) + \ell(\bar{\Delta}) - 2}(\Delta, \bar{\Delta})}{\beta^2 (\ell(\Delta) + \ell(\bar{\Delta}) - 2)!} \prod_{i=1}^{\ell(\Delta)} (\beta \mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\Delta})} (\beta \bar{\mu}_i \bar{t}_{\bar{\mu}_i})$$

is the generating function for the numbers of the ramified coverings $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$.

In [25] Okounkov has proved that the (dispersionfull) tau-function

$$\tau_n(\mathbf{t}, \bar{\mathbf{t}}) = e^{\frac{1}{12} \beta n(n+1)(2n+1)} Q^{\frac{1}{2} n(n+1)} \exp\left(\Phi(\beta, e^{\beta(n+\frac{1}{2})} Q, \mathbf{t}, \bar{\mathbf{t}})\right)$$

solves the 2DTL hierarchy of Ueno and Takasaki [32]. The dispersionless limit was characterized in terms of string equations by Takasaki [27]. The procedure of passing to the dispersionless limit [29] is equivalent to extracting the contribution of genus 0 surfaces. Thus the function

$$(12) \quad \begin{aligned} F(\beta, Q, t_0, \mathbf{t}, \bar{\mathbf{t}}) &= \frac{\beta t_0^3}{6} + \frac{t_0^2}{2} \log Q + \Phi_0(\beta, Q e^{\beta t_0}, \mathbf{t}, \bar{\mathbf{t}}) \\ &= \frac{\beta t_0^3}{6} + \frac{t_0^2}{2} \log Q + \sum_{d \geq 1} \sum_{|\Delta| = |\bar{\Delta}| = d} \frac{(Q e^{\beta t_0})^d H_{d, \ell(\Delta) + \ell(\bar{\Delta}) - 2}(\Delta, \bar{\Delta})}{\beta^2 (\ell(\Delta) + \ell(\bar{\Delta}) - 2)!} \prod_{i=1}^{\ell(\Delta)} (\beta \mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\Delta})} (\beta \bar{\mu}_i \bar{t}_{\bar{\mu}_i}) \end{aligned}$$

satisfies the dispersionless 2DTL hierarchy from the previous section. Moreover, from the structure of the series we see that it is a symmetric solution.

3.2. Formulas for double Hurwitz numbers. Now we can use Theorem 2.1 to calculate the genus 0 double Hurwitz numbers. Note that at genus 0 the Hurwitz number $H_{d,l}(\Delta|\bar{\Delta})$ depends on the diagrams $\Delta, \bar{\Delta}$ only (the numbers d, l are restored as $d = |\Delta| = |\bar{\Delta}|$, $l = \ell(\Delta) + \ell(\bar{\Delta}) - 2$). We will write $H_0(\Delta|\bar{\Delta}) = H_{d,l}(\Delta|\bar{\Delta})$.

Theorem 3.1. *The genus 0 double Hurwitz number $H_0(\Delta|\bar{\Delta})$ is given by*

$$H_0(\Delta|\bar{\Delta}) = \frac{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!}{\rho(\Delta) \rho(\bar{\Delta})} \sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$$

where the sum is carried over all matrices $\begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$ such that $s_1 + \dots + s_m = |\Delta|$ and $r_1 + \dots + r_m = m + \ell(\Delta) + \ell(\bar{\Delta}) - 2$

Proof. The function

$$F(1, 1, t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta| = |\bar{\Delta}|} \frac{e^{|\Delta| t_0} H_0(\Delta, \bar{\Delta})}{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!} \prod_{i=1}^{\ell(\Delta)} (\mu_i t_{\mu_i}) \prod_{i=1}^{\ell(\bar{\Delta})} (\bar{\mu}_i \bar{t}_{\bar{\mu}_i})$$

is a symmetric solution of the dispersionless 2DTL hierarchy with $f(t_0) = e^{t_0}$. Therefore, according to Theorem 2.1,

$$F(1, 1, t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta| = |\bar{\Delta}|} \sum_{\substack{s_1 + \dots + s_m = |\Delta| \\ r_1 + \dots + r_m = m + \ell(\Delta) + \ell(\bar{\Delta}) - 2}} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix} \partial_0^{r_1} (e^{s_1 t_0}) \dots \partial_0^{r_m} (e^{s_m t_0}) t_{\Delta} t_{\bar{\Delta}}$$

Comparing this with the previous formula, we find that

$$\rho(\Delta)\rho(\bar{\Delta})\frac{H_0(\Delta, \bar{\Delta})}{(\ell(\Delta)+\ell(\bar{\Delta})-2)!} = \sum_{\substack{s_1+\dots+s_m=|\Delta| \\ r_1+\dots+r_m=m+\ell(\Delta)+\ell(\bar{\Delta})-2}} N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m} s_1^{r_1} \dots s_m^{r_m}$$

which is the assertion of the theorem. \square

Remark. Theorem 3.1 and the geometric definition of Hurwitz numbers give some nontrivial combinatorial relations. First, the sum $\sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$ must be positive despite the fact that the numbers $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$ may have different signs. Second, the equality $H_0(\Delta|\bar{\Delta}) = H_0(\bar{\Delta}|\Delta)$ implies the identity

$$\sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m} = \sum s_1^{r_1} \dots s_m^{r_m} N_{(\bar{\Delta}|\Delta)} \binom{s_1 \dots s_m}{r_1 \dots r_m}$$

which is non-trivial because all the definitions depend on the order of the Young diagrams.

It follows from Theorem 3.1 that the formulas for $N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m}$ give explicit expressions for the genus 0 double Hurwitz numbers. Let us consider some examples.

Corollary 3.1. *The number of polynomials of degree n with a single critical value at 0 of the type Δ is*

$$H_0(\Delta|[n]) = \frac{(\ell(\Delta)-1)!}{\sigma(\Delta)} n^{\ell(\Delta)-2}.$$

Proof. It follows from our definition that $N_{(\Delta|[n])} \binom{r_1 \dots r_m}{n_1 \dots n_m} = 0$ for $m > 1$. Hence, according to Theorem 3.1 and Lemma 2.3,

$$H_0(\Delta|[n]) = \frac{(\ell(\Delta)-1)!}{\rho(\Delta)\rho([n])} |\Delta|^{\ell(\Delta)-1} \frac{\rho(\Delta)\rho([n])}{|\Delta|\sigma(\Delta)} = \frac{(\ell(\Delta)-1)!}{\sigma(\Delta)} n^{\ell(\Delta)-2}.$$

\square

Corollary 3.2. *Let $i_1 + i_2 = \bar{i}_1 + \bar{i}_2 = d$. Then*

$$H_0([i_1, i_2]|\bar{i}_1, \bar{i}_2) = 2 \frac{d - \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\}}{(1 + \delta_{i_1 i_2})(1 + \delta_{\bar{i}_1 \bar{i}_2})}$$

Proof. Put $\Delta = [i_1, i_2]$, $\bar{\Delta} = [\bar{i}_1, \bar{i}_2]$. It follows from our definition that $N_{(\Delta|\bar{\Delta})} \binom{r_1 \dots r_m}{n_1 \dots n_m} = 0$ for $m > 2$. According to Theorem 3.1 and Lemmas 2.3, 2.4 we thus have:

$$\begin{aligned} H_0(\Delta|\bar{\Delta}) &= \frac{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!}{\rho(\Delta)\rho(\bar{\Delta})} \sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \binom{s_1 \dots s_m}{r_1 \dots r_m} \\ &= \frac{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!}{\rho(\Delta)\rho(\bar{\Delta})} \left(|\Delta|^{\ell(\Delta)+\ell(\bar{\Delta})-2} N_{(\Delta|\bar{\Delta})} \binom{|\Delta|}{\ell(\Delta) + \ell(\bar{\Delta}) - 2} + 2\bar{i}_1\bar{i}_2 \tilde{N}_{\binom{i_1 i_2}{\bar{i}_1 \bar{i}_2}} \binom{\bar{i}_1 \bar{i}_2}{11} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!}{\rho(\Delta)\rho(\bar{\Delta})} \left(|\Delta|^{\ell(\Delta) + \ell(\bar{\Delta}) - 2} \frac{\rho(\Delta)\rho(\bar{\Delta})}{|\Delta|\sigma(\Delta)\sigma(\bar{\Delta})} - \frac{i_1 i_2 \bar{i}_1 \bar{i}_2}{\sigma(\Delta)\sigma(\bar{\Delta})} \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\} \right) \\
&= \frac{(\ell(\Delta) + \ell(\bar{\Delta}) - 2)!}{\sigma(\Delta)\sigma(\bar{\Delta})} \left(|\Delta|^{\ell(\Delta) + \ell(\bar{\Delta}) - 3} - \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\} \right) = 2 \frac{|\Delta| - \min\{i_1, i_2, \bar{i}_1, \bar{i}_2\}}{\sigma(\Delta)\sigma(\bar{\Delta})}
\end{aligned}$$

□

4. CONFORMAL DYNAMICS

The result of section 2 takes on a geometric significance when the Toda times t_k, \bar{t}_k are identified with (complex conjugate) moments of simply-connected domains in the complex plane with smooth boundary. In this case the dispersionless 2DTL dynamics encodes the shape dependence of the conformal mapping of such a domain to some fixed reference domain, thus providing an effectivization of the Riemann mapping theorem. We call it the *conformal dynamics*.

4.1. Local coordinates in the space of simply-connected domains. Let $D \subset \mathbb{C}$ be a compact simply-connected domain whose boundary is a smooth curve $\gamma = \partial D$ and let $D^c = \hat{\mathbb{C}} \setminus D$ be its complement in the Riemann sphere $\hat{\mathbb{C}}$. Without loss of generality, it is convenient to assume that $D \ni 0$ (and so $D^c \ni \infty$).

Fix a real-analytic and real-valued function $U(z, \bar{z})$ in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that

$$\sigma(z, \bar{z}) := \partial \bar{\partial} U(z, \bar{z}) > 0.$$

We introduce the set of moments of the domain D^c as follows:

$$t_k = \frac{1}{2\pi i k} \oint_{\gamma} z^{-k} \partial U(z, \bar{z}) dz = -\frac{1}{\pi k} \iint_{D^c} z^{-k} \sigma(z, \bar{z}) d^2 z, \quad k \geq 1,$$

where $d^2 z \equiv dx dy$. In general they are complex numbers. We claim that together with the real parameter

$$t_0 = \frac{1}{2\pi i} \oint_{\gamma} \partial U(z, \bar{z}) dz = \frac{1}{\pi} \iint_D \sigma(z, \bar{z}) d^2 z$$

they form a set of local coordinates in the space \mathcal{H} of such domains D (or D^c). The function σ plays the role of a background charge density in the complex plane.

First we prove the local uniqueness of a domain with given moments [4, 15].

Proposition 4.1. *Any one-parameter deformation $D(t)$ of $D = D(0)$ with some real parameter t such that all t_k are preserved, $\partial_t t_k = 0$, $k \geq 0$, is trivial.*

Proof. The proof is a modification of the one presented in [15] for the case $\sigma(z, \bar{z}) = 1$. It is based on the following two facts.

- The difference of the boundary values $\partial_t C^\pm(\zeta) d\zeta$ of the t -derivative of the Cauchy integral

$$C(z) dz = \frac{dz}{2\pi i} \oint_{\gamma} \frac{U_\zeta(\zeta, \bar{\zeta}) d\zeta}{\zeta - z}, \quad U_\zeta(\zeta, \bar{\zeta}) \equiv \partial_\zeta U(\zeta, \bar{\zeta}),$$

is a purely imaginary differential on γ . Indeed, let $\zeta(\theta, t)$ be a parameterization of the curve $\gamma(t)$. Taking the t -derivative of the Cauchy integral and integrating by parts, one gets

$$\begin{aligned}\partial_t C(z)dz &= \frac{dz}{2\pi i} \oint_{\gamma} \left(\frac{U_{\zeta\zeta}\zeta_t\bar{\zeta}_{\theta} + U_{\zeta\bar{\zeta}}\bar{\zeta}_t\zeta_{\theta} + U_{\zeta}\zeta_{t\theta}}{\zeta - \theta} - \frac{U_{\zeta}\zeta_t\zeta_{\theta}}{(\zeta - z)^2} \right) d\theta \\ &= \frac{dz}{2\pi i} \oint_{\gamma} \frac{\bar{\zeta}_t\zeta_{\theta} - \zeta_t\bar{\zeta}_{\theta}}{\zeta - z} \sigma(\zeta(\theta, t), \bar{\zeta}(\theta, t)) d\theta\end{aligned}$$

Hence, $\left(\partial_t C^+(\zeta) - \partial_t C^-(\zeta)\right)d\zeta = \sigma(\zeta, \bar{\zeta})(\partial_t \bar{\zeta}d\zeta - \partial_t \zeta d\bar{\zeta}) = 2i\sigma \operatorname{Im}(\partial_t \bar{\zeta}d\zeta)$ is indeed purely imaginary.

- If a t -deformation preserves all the moments t_k , $k \geq 0$, then the differential $\sigma(\zeta, \bar{\zeta})(\partial_t \bar{\zeta}d\zeta - \partial_t \zeta d\bar{\zeta})$ extends to a holomorphic differential in D^c . Take a small neighbourhood B of $0 \in D$ such that $|z| < |\zeta|$ for all $z \in B$ and $\zeta \in \gamma$, then for $z \in B$ we can expand:

$$\partial_t C^+(z)dz = \frac{\partial}{\partial t} \left(\frac{dz}{2\pi i} \sum_{k \geq 0} z^k \oint_{\gamma} \zeta^{-k-1} U_{\zeta}(\zeta, \bar{\zeta}) d\zeta \right) = \sum_{k \geq 1} k(\partial_t t_k) z^{k-1} dz = 0$$

and, since C^+ is analytic in D , we conclude that $\partial_t C^+ \equiv 0$. Therefore, the differential $\sigma(\zeta, \bar{\zeta})(\partial_t \bar{\zeta}d\zeta - \partial_t \zeta d\bar{\zeta})$ is the boundary value of the differential $-\partial_t C^-(z)dz$ which has at most simple pole at ∞ and holomorphic everywhere else in D^c . The equality

$$\partial_t t_0 = \frac{1}{2\pi i} \oint_{\gamma} \sigma(\zeta, \bar{\zeta})(\partial_t \bar{\zeta}d\zeta - \partial_t \zeta d\bar{\zeta}) = 0$$

then implies that the residue at ∞ vanishes, so $\partial_t C^-(z)dz$ is a holomorphic differential.

Any holomorphic differential which is purely imaginary along the boundary of a simply-connected domain must identically vanish in this domain. Hence we conclude that $\partial_t \bar{\zeta}d\zeta - \partial_t \zeta d\bar{\zeta} = 0$ which means that the vector $\partial_t \zeta$ is tangent to the boundary and hence the deformation is trivial. \square

In fact the assertion of the Proposition remains true under less restrictive conditions on the function σ : it is enough to require that $\sigma \neq 0$ in some strip-like neighbourhood of the contour γ .

The fact that the set of the moments is not overcomplete follows from the explicit construction of vector fields in the space of domains that change real or imaginary part of any moment keeping all the others fixed (see below). These arguments allow one to prove the following theorem.

Theorem 4.1. *The real parameters t_0 , $\operatorname{Re} t_k$, $\operatorname{Im} t_k$, $k \geq 1$, form a set of local coordinates in the space of simply-connected plane domains with smooth boundary.*

This statement allows one to identify functionals on the space of domains D with functions of infinitely many independent variables $t_0, \{t_k\}, \{\bar{t}_k\}$.

4.2. The Green function of the Dirichlet boundary value problem and special deformations. According to the Riemann mapping theorem, there exists a conformal bijection $w : D^c \rightarrow U$ between D^c and the exterior of the unit disk $U = \{u \in \hat{\mathbb{C}} \mid |u| > 1\}$. If the conformal map $w(z)$ is known, one can construct the Green function of the domain D^c ,

$$(13) \quad G(z, \xi) = \frac{1}{2\pi} \log \left| \frac{w(z) - w(\xi)}{w(z)\overline{w(\xi)} - 1} \right|,$$

which solves the Dirichlet boundary value problem in D^c . To wit, the Poisson formula

$$(14) \quad u(z) = - \oint_{\gamma} u_0(\xi) \partial_{n_\xi} G(z, \xi) |d\xi|$$

restores a harmonic function in D^c from its boundary value $u_0 = u|_{\gamma}$. Here ∂_{n_ξ} denotes the derivative along the outward normal vector to the boundary of D with respect to the second variable and $|d\xi|$ is an infinitesimal element of length along the boundary. The Green function $G(z, \xi) = \frac{1}{2\pi} \log |z - \xi| + g(z, \xi)$ in $D^c \times D^c$ is uniquely defined by the following properties [10]:

- $G(z, \xi) = G(\xi, z)$ and $G(z, \xi') = 0$ for any $z \in D^c$ and $\xi' \in \gamma$;
- The function $g(z, \xi)$ is continuous in $D^c \times D^c$ and is continuous for ξ in the closure of D^c at any $z \in D^c$;
- The function $g(z, \xi)$ is harmonic in z for any $\xi \in D^c$ and is harmonic in ξ for any $z \in D^c$.

Let us introduce the differential operator

$$(15) \quad \nabla(z) = \partial_0 + D(z) + \bar{D}(\bar{z}),$$

where $D(z), \bar{D}(\bar{z})$ are given by (4).

Fix a point $a \in D^c$ and consider a special infinitesimal deformation of the form

$$(16) \quad \delta_a n(z) = -\frac{\varepsilon\pi}{\sigma(z, \bar{z})} \partial_{n_z} G(a, z), \quad z \in \gamma, \quad \varepsilon \rightarrow 0,$$

where $\delta_a n(z)$ is the normal displacement of the boundary (positive if directed outward D) at the boundary point (equivalently, one may speak about the normal “velocity” of the boundary deformation which is $V_a(z) = \lim_{\varepsilon \rightarrow 0} (\delta n_a(z)/\varepsilon)$). For any sufficiently smooth initial boundary this deformation is well-defined as $\varepsilon \rightarrow 0$. By δ_a we denote the variation of any quantity under this deformation.

Lemma 4.1. *Let X be any functional on the space of domains D regarded as a function of $t_0, \{t_k\}, \{\bar{t}_k\}$, then for any $z \in D^c$ we have $\delta_z X = \varepsilon \nabla(z) X$.*

Proof. It is easy to see that

$$\delta_z t_0 = -\frac{\varepsilon}{2\pi} \oint_{\gamma} \partial_{n_\xi} G(z, \xi) |d\xi| = \varepsilon, \quad \delta_z t_k = -\frac{\varepsilon}{2\pi k} \oint_{\gamma} \xi^{-k} \partial_{n_\xi} G(z, \xi) |d\xi| = \frac{\varepsilon}{k} z^{-k}$$

by virtue of the Poisson formula (14). Therefore by Theorem 4.1 we have:

$$\delta_z X = \frac{\partial X}{\partial t_0} \delta_z t_0 + \sum_{k \geq 1} \frac{\partial X}{\partial t_k} \delta_z t_k + \sum_{k \geq 1} \frac{\partial X}{\partial \bar{t}_k} \delta_z \bar{t}_k = \varepsilon \left(\partial_0 + \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k + \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k \right) X.$$

□

Now we can explicitly define the deformations that change only either $x_k = \operatorname{Re} t_k$ or $y_k = \operatorname{Im} t_k$ keeping all other moments fixed. From the proof of Lemma 4.1 it follows that the normal displacements $\delta n(\xi) = \varepsilon \operatorname{Re}(\partial_{n_\xi} H_k(\xi))$ and $\delta n(\xi) = \varepsilon \operatorname{Im}(\partial_{n_\xi} H_k(\xi))$, where

$$H_k(\xi) = -i \oint_{\infty} z^k \partial_z G(z, \xi) dz$$

(the contour integral goes around infinity) change the real and imaginary parts of t_k by $\pm \varepsilon$ respectively keeping all other moments unchanged. In particular, the deformation

$$\delta_{\infty} n(\xi) = -\frac{\varepsilon \pi}{\sigma(\xi, \bar{\xi})} \partial_{n_\xi} G(\infty, \xi)$$

changes t_0 only. Therefore, the vector fields $\partial/\partial t_0$, $\partial/\partial x_k$, $\partial/\partial y_k$ in the space of domains are locally well-defined and commute. Existence of such vector fields means that the variables t_k are independent and $\partial_k = \frac{1}{2}(\partial_{x_k} - i\partial_{y_k})$, $\bar{\partial}_k = \frac{1}{2}(\partial_{x_k} + i\partial_{y_k})$ can be understood as partial derivatives.

We will also need the following simple lemma.

Lemma 4.2. *Let X be a functional of the form $X = \int_D \Psi(\zeta, \bar{\zeta}) \sigma(\zeta, \bar{\zeta}) d^2 \zeta$ with an arbitrary integrable function Ψ regular on the boundary, then*

$$\nabla(z)X = \pi \Psi^H(z),$$

where $\Psi^H(z)$ is the (unique) harmonic extension of the function Ψ from the boundary to the domain D^c .

Proof. The variation of X under the special deformation (16) is

$$\delta_z X = \oint_{\gamma} \Psi(\zeta, \bar{\zeta}) \sigma(\zeta, \bar{\zeta}) \delta n_z(\zeta) |d\zeta| = -\varepsilon \pi \oint_{\gamma} \Psi(\zeta, \bar{\zeta}) \partial_{n_\zeta} G(z, \zeta) |d\zeta|$$

Now the assertion obviously follows from Lemma 4.1 and the Poisson formula (14). \square

4.3. The dispersionless tau-function. Consider the following functional on the space of domains D :

$$(17) \quad F = -\frac{1}{\pi^2} \iint_D \iint_D \sigma(z, \bar{z}) \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) d^2 z d^2 \zeta.$$

Theorem 4.2. ([31, 12, 18, 35]) *It holds*

$$(18) \quad G(z, \zeta) = \frac{1}{2\pi} \log |z^{-1} - \zeta^{-1}| + \frac{1}{4\pi} \nabla(z) \nabla(\zeta) F.$$

Proof. The proof consists in successive application of Lemma 4.2 and using the characteristic properties of the Green function. Applying Lemma 4.2 to (17), we get:

$$(19) \quad \nabla(z)F = -\frac{2}{\pi} \iint_D \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) d^2 \zeta$$

for $z \in D^c$. Applying the lemma once again, we conclude that $\nabla(\zeta) \nabla(z)F$ is the harmonic continuation of the function $-2 \log |z^{-1} - \zeta^{-1}|$ from the boundary to the domain D^c . This function is harmonic everywhere in D^c except at $\zeta = z$, where it has the logarithmic singularity. It can be cancelled, without changing the boundary value, by adding the function $4\pi G(z, \zeta)$. \square

Let $w(z)$ be the conformal map from D^c onto the exterior of the unit circle normalized by the conditions $w(\infty) = \infty$ and $w'(\infty)$ is real positive. It has the form $w(z) = pz + \sum_{j \geq 0} p_j z^{-j}$, where $p > 0$.

Corollary 4.1. *The conformal map $w(z)$ is given by*

$$(20) \quad w(z) = z \exp \left(\left(-\frac{1}{2} \partial_0^2 - \partial_0 D(z) \right) F \right).$$

Proof. From equation (13) it follows that $2\pi G(z, \infty) = -\log |w(z)|$. Tending $\xi \rightarrow \infty$ in (18) and separating holomorphic and antiholomorphic parts in z , we get the result. \square

Note that the limit $z \rightarrow \infty$ in (20) yields $\log p = -\frac{1}{2} \partial_0^2 F$.

The following theorem establishes the embedding of the conformal dynamics into the dispersionless 2DTL hierarchy and identifies F with the dispersionless tau-function.

Theorem 4.3. ([15, 35]) *The function F satisfies the equations*

$$\begin{aligned} (z - \xi) e^{D(z)D(\xi)F} &= z e^{-\partial_0 D(z)F} - \xi e^{-\partial_0 D(\xi)F}, \\ (\bar{z} - \bar{\xi}) e^{\bar{D}(\bar{z})\bar{D}(\bar{\xi})F} &= \bar{z} e^{-\partial_0 \bar{D}(\bar{z})F} - \bar{\xi} e^{-\partial_0 \bar{D}(\bar{\xi})F}, \\ 1 - e^{-D(z)\bar{D}(\bar{\xi})F} &= \frac{1}{z\bar{\xi}} e^{\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F} \end{aligned}$$

(cf. (1)-(3)).

Proof. The proof is the same as in [15]. Combining (13) and (18), we get

$$\log \left| \frac{w(z) - w(\xi)}{1 - w(z)\bar{w}(\xi)} \right| = \log \left| \frac{1}{z} - \frac{1}{\xi} \right| + \frac{1}{2} \nabla(z) \nabla(\xi) F.$$

Next, substituting here $w(z)$ from (20) and separating holomorphic and antiholomorphic parts in z, ξ , we obtain the desired result. \square

Important remark. Note that although the definitions of the moments and the function F essentially depend on the background density σ , the formulas for the Green function and the conformal map (18), (20) are σ -independent. This means that the conformal dynamics is described by any solution to the dispersionless 2DTL hierarchy of this class.

4.4. Complimentary moments. The integral of Cauchy type

$$C(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{U_{\zeta}(\zeta, \bar{\zeta}) d\zeta}{\zeta - z}, \quad U_{\zeta}(\zeta, \bar{\zeta}) \equiv \partial_{\zeta} U(\zeta, \bar{\zeta}),$$

defines a function which is analytic in D and D^c with a jump across γ . Let $C^{\pm}(z)$ be the analytic functions defined by this integral in D and D^c respectively. By the Sokhotski-Plemelj formula, the jump of the function $C(z)$ across the contour γ is equal to $\partial U(z, \bar{z})$:

$$(21) \quad (C^+(z) - C^-(z))|_{z \in \gamma} = \partial U(z, \bar{z}).$$

As it follows from the proof of Proposition 4.1, $C^+(z)$ is the generating function of the moments t_k :

$$C^+(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{U_{\zeta}(\zeta, \bar{\zeta}) d\zeta}{\zeta - z} = \sum_{k \geq 1} k t_k z^{k-1}, \quad z \in D.$$

Similarly, $C^-(z)$ is the generating function of the set of *complimentary moments* v_k :

$$C^-(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{U_{\zeta}(\zeta, \bar{\zeta}) d\zeta}{\zeta - z} = -\frac{t_0}{z} - \sum_{k \geq 1} v_k z^{-k-1}, \quad z \in \mathbb{D}^c,$$

which are given by the integrals

$$v_k = \frac{1}{2\pi i} \oint_{\gamma} z^k \partial U(z, \bar{z}) dz = \frac{1}{\pi} \iint_{\mathbb{D}} z^k \sigma(z, \bar{z}) d^2 z, \quad k \geq 1.$$

They are functions of the moments $t_0, \{t_k\}, \{\bar{t}_k\}$. It is also useful to introduce the logarithmic moment

$$v_0 = \frac{1}{\pi} \iint_{\mathbb{D}} \log |z|^2 \sigma(z, \bar{z}) d^2 z.$$

Theorem 4.4. (cf. [18, 33, 35]) *The following relations hold:*

$$(22) \quad v_0 = \partial_0 F, \quad v_k = \partial_k F, \quad \bar{v}_k = \bar{\partial}_k F, \quad k \geq 1.$$

Proof. Applying Lemma 4.2 to (17), we get (19). The expansion of both sides in powers of z, \bar{z} yields (22). \square

Proposition 4.2. *Suppose that only a finite number of the moments t_k are different from 0. Then the tau-function (17) can be represented as*

$$(23) \quad 2F = -\frac{1}{\pi} \iint_{\mathbb{D}} U \sigma d^2 z + t_0 v_0 + \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k).$$

Proof. Set $\phi(z, \bar{z}) = -\frac{1}{\pi} \iint_{\mathbb{D}} \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) d^2 \zeta$, then under the assumption of the proposition we have the expansion

$$\phi(z, \bar{z}) = -U(z, \bar{z}) + t_0 \log |z|^2 + \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k)$$

valid everywhere in $\mathbb{D} \setminus \{0\}$. Substituting this into (17) written in the form $2F = \frac{1}{\pi} \iint_{\mathbb{D}} \phi \sigma d^2 z$, performing the termwise integration and using the definition of the complimentary moments, we get (23). \square

5. CONFORMAL DYNAMICS IN A SYMMETRIC BACKGROUND

5.1. The general axially symmetric case. The case when the background density function σ (and the function U) is *axially symmetric*, i.e., depends only on $|z|$, is of a special interest. Note that for any axially symmetric background it holds $z \partial U = \bar{z} \bar{\partial} U$. We denote the symmetric density by $\sigma(|z|^2)$. In this case the conformal dynamics provides a symmetric solution to the dispersionless 2DTL, so the function F has the explicit Taylor expansion obtained in section 2. Indeed, the symmetry implies that when all moments t_k at $k \geq 1$ are equal to 0, the domain \mathbb{D} is just a disk of radius R such that

$$(24) \quad t_0 = \int_0^{R^2} \sigma(x) dx, \quad v_0 = \int_0^{R^2} \log x \sigma(x) dx$$

and the complimentary moments v_k with $k \geq 1$ are zero. From (22) we see that

$$(25) \quad v_k|_{t_0} = \partial_{t_k} F|_{t_0} = 0 \quad \text{for all } k \geq 1$$

so the solution is indeed symmetric in the sense of section 2. One can see that $\partial v_0 / \partial t_0 = \log R^2$, i.e., $p = 1/R$ as it should be. We also have

$$(26) \quad F|_{t_0} = \int_0^{R^2} \sigma(y) dy \int_0^y \log x \sigma(x) dx, \quad f(t_0) = e^{F|_{t_0}''} = R^2(t_0),$$

where R^2 should be understood as a function of t_0 implicitly given by (24).

The Green function at $t_k = 0$ is

$$2\pi G(z, \zeta) = \log \left| \frac{R^2(z - \zeta)}{z\bar{\zeta} - R^2} \right| = \log |z^{-1} - \zeta^{-1}| + \log R - \operatorname{Re} \sum_{k \geq 1} \frac{R^{2k}}{k z^k \bar{\zeta}^k}$$

Comparison with (18) yields

$$(27) \quad \partial_{t_j} \partial_{t_k} F|_{t_0} = 0, \quad \partial_{t_j} \partial_{t_k} F|_{t_0} = k R^{2k} \delta_{jk}$$

in accordance with Lemma 2.1.

Note that in the axially symmetric case the moment v_0 can be represented as a contour integral:

$$v_0 = \frac{1}{2\pi i} \oint_{\gamma} (\log |z|^2 \partial U - z^{-1} U) dz.$$

5.2. Examples of symmetric solutions.

5.2.1. *The homogeneous density.* An important example is the homogeneous density

$$(28) \quad \sigma(z, \bar{z}) = (z\bar{z})^{\alpha-1}, \quad U(z, \bar{z}) = \frac{1}{\alpha^2} (z\bar{z})^{\alpha}$$

with some $\alpha \in \mathbb{R}$. Assume that $\alpha > 0$. In this case $t_0 = R^{2\alpha}/\alpha$, so $f(t_0) = (\alpha t_0)^{1/\alpha}$ and

$$F|_{t_0} = \frac{1}{4\alpha^3} (f^{2\alpha} \log f^{2\alpha} - 3f^{2\alpha}) = \frac{t_0^2}{2\alpha} \log(\alpha t_0) - \frac{3t_0^2}{4\alpha}$$

(recall that for symmetric density $f = R^2$).

Proposition 5.1. *Assuming that only a finite number of the moments t_k are different from 0, the dispersionless tau-function for this solution is quasi-homogeneous, that is it obeys the relation*

$$(29) \quad 4\alpha F = -t_0^2 + 2\alpha t_0 \partial_0 F + \sum_{k \geq 1} (2\alpha - k) (t_k \partial_k F + \bar{t}_k \bar{\partial}_k F).$$

Proof. We use equation (23). In the integral term we write $U\sigma = \bar{\partial}(U\partial U) - \partial U \bar{\partial} U$ and notice that for the particular function U we have $U\sigma = \partial U \bar{\partial} U$, and also $z\partial U = \alpha U$, so $U\sigma = \frac{1}{2} \bar{\partial}(U\partial U)$. This allows us to transform the 2D integral to a contour integral:

$$\frac{1}{\pi} \iint_{\mathbb{D}} U \sigma d^2 z = \frac{1}{4\pi i \alpha} \oint_{\gamma} (z\partial U)^2 \frac{dz}{z}.$$

Now recall (21) and represent $\partial U = C^+ - C^-$ (on γ), with C^+ being a polynomial. Shrinking the integration contour to ∞ , we obtain:

$$\frac{1}{2\pi i} \oint_{\gamma} (z\partial U)^2 \frac{dz}{z} = t_0^2 + 2 \sum_{k \geq 1} k t_k v_k.$$

Since the initial integral is obviously real, we conclude that $\sum_{k \geq 1} k t_k v_k$ is a real quantity. The quasi-homogeneity relation (29) follows. \square

Remark. The case $\alpha = 1$ ($\sigma(z, \bar{z}) = 1$) was considered in [18, 33]. The Taylor coefficients for this solution were found in [22]. In this case the conformal dynamics is identical to the Darcy law for the motion of interface between viscous and non-viscous fluids confined in the radial Hele-Shaw cell, assuming that there is no surface tension at the interface. The normal velocity of the interface γ is proportional to the gradient of the Green function of the Laplace operator. This type of processes is known as *Laplacian growth* (see, e.g., [7, 20]). As is mentioned in [21], the conformal dynamics in the case $\alpha = 1/N$ for an integer $N > 0$ can be mapped to a Laplacian growth process in a sector with angle $2\pi/N$ and periodic conditions at the boundary rays (a cone). With proper modifications, the case $\alpha < 0$ (and, in particular, $\alpha = -1$) can be also considered and mapped to the Laplacian growth in the compact *interior* domain bounded by the curve γ .

5.2.2. *The solution yielding the Hurwitz numbers.* In this case

$$(30) \quad \sigma(z, \bar{z}) = \frac{1}{\beta z \bar{z}}, \quad U(z, \bar{z}) = \frac{1}{2\beta} \left[\log \frac{z \bar{z}}{Q} \right]^2$$

which can be formally regarded as a limit $\alpha \rightarrow 0$ of (28) but the limit is tricky and this case deserves a separate consideration. In this case

$$F|_{t_0} = \frac{\beta t_0^3}{6} + \frac{t_0^2}{2} \log Q, \quad f(t_0) = Q e^{\beta t_0}.$$

Remark. As is shown in [34], the conformal dynamics in this case can be mapped to a Laplacian growth process in an infinite channel with periodic conditions in the transverse direction, i.e., on the surface of an infinite cylinder of radius $1/\beta$.

Proposition 5.2. ([34]) *The dispersionless tau-function for this solution obeys the following homogeneity relation:*

$$2F = \tau \partial_\tau F + t_0 \partial_0 F + \sum_{k \geq 1} (t_k \partial_k F + \bar{t}_k \bar{\partial}_k F), \quad \tau \equiv 1/\beta.$$

Proof. We have $t_k = \frac{1}{2\pi i k \beta} \oint_\gamma z^{-k} \log\left(\frac{|z|^2}{Q}\right) \frac{dz}{z}$, $k \geq 1$, $t_0 = \frac{1}{2\pi i \beta} \oint_\gamma \log\left(\frac{|z|^2}{Q}\right) \frac{dz}{z}$, so that the variables $\hat{t}_k = \beta t_k$ are β -independent. As is seen from equation (17) with σ as in (30), F is of the form $F = \beta^{-2} \hat{F}$, where \hat{F} is β -independent. Therefore,

$$F(\beta, \{t_k\}) = F(\beta, \{\hat{t}_k/\beta\}) = \beta^{-2} \hat{F} = \beta^{-2} F(1, \{\hat{t}_k\}).$$

Taking the total β -derivative of the identity $\beta^2 F(\beta, \{\hat{t}_k/\beta\}) = F(1, \{\hat{t}_k\})$, we get:

$$-2\beta^3 F - \beta^4 \partial_\beta F + \beta^2 \left(\hat{t}_0 \partial_0 F + 2 \operatorname{Re} \sum_{k \geq 1} \hat{t}_k \partial_k F \right) = 0$$

or

$$2F = -\beta \partial_\beta F + t_0 \partial_0 F + \sum_{k \geq 1} (t_k \partial_k F + \bar{t}_k \bar{\partial}_k F).$$

□

The Taylor expansion of this function is given by (12), with the Taylor coefficients being essentially the double Hurwitz numbers for connected genus 0 coverings. The homogeneity property can be explicitly seen from the Taylor expansion.

Proposition 5.3. *The dispersionless tau-function for the solution determined by the data (30) satisfies the relations*

$$(31) \quad \begin{aligned} \frac{\partial F}{\partial \log Q} &= \frac{t_0^2}{2} + \sum_{k \geq 1} k t_k \partial_k F, \\ \frac{\partial F}{\partial \beta} &= \frac{t_0^3}{6} + t_0 \sum_{k \geq 1} k t_k \partial_k F + \frac{1}{2} \sum_{k, l \geq 1} (k l t_k t_l \partial_{k+l} F + (k+l) t_{k+l} \partial_k F \partial_l F). \end{aligned}$$

Proof. The first formula is obvious from the structure of the Taylor expansion (12). Another proof, similar to that of Proposition 5.1, is given in [34]. For the proof of the second formula we note that in the case (30) $U\sigma = \frac{1}{3} \bar{\partial}(U\partial U)$ and thus

$$\frac{1}{\pi} \iint_{\mathbb{D}} U \sigma \, d^2 z = \frac{\beta}{12\pi i} \oint_{\gamma} (z \partial U)^3 \frac{dz}{z}.$$

Proceeding as in the proof of Proposition 5.1, we obtain

$$\frac{1}{12\pi i} \oint_{\gamma} (z \partial U)^3 \frac{dz}{z} = \frac{t_0^3}{6} + t_0 \sum_{k \geq 1} k t_k v_k + \frac{1}{2} \sum_{k, l \geq 1} (k l t_k t_l v_{k+l} + (k+l) t_{k+l} v_k v_l).$$

Then from Proposition 4.2 we have:

$$(32) \quad \begin{aligned} 2F &= t_0 v_0 + \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k) - \frac{\beta t_0^3}{6} - \beta t_0 \sum_{k \geq 1} k t_k v_k \\ &\quad - \frac{\beta}{2} \sum_{k, l \geq 1} (k l t_k t_l v_{k+l} + (k+l) t_{k+l} v_k v_l). \end{aligned}$$

Combining this with the homogeneity property (Proposition 5.2), we arrive at the second formula in (31). \square

Remark. The double sum in the second equation in (31) is the genus 0 part of the celebrated cut-and-join operator [5], see also [27].

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